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III*—ARISTOTLE, ZENO, AND THE POTENTIAL INFINITE

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I ARISTOTLE ON THE POTENTIAL INFINITE

Aristotle is generally thought to have held that nothing is *actually* infinite though several things are *potentially* infinite. It is certainly true that Aristotle often expresses his view in just this way, but it seems also to be true that he recognised that this was to some extent a misleading way of expressing it. Consider, for instance, the words with which he first introduces the view, in *Physics* III.

In chapter 5 he has been arguing that there cannot be any physical body which is infinitely large, though he expresses this simply as the claim that no physical body is, or could be, infinite. In chapter 6, however, he observes that we can hardly deny the infinite any existence at all, for that would imply that time had a beginning and an end, that magnitudes were not always divisible into further magnitudes, and that the number series came to a stop. From this he infers that we need to distinguish the senses in which the infinite may be said to exist or not to exist, and continues:

> A thing may be said to be either potentially or actually, and a thing may be infinite either by addition or by division. Now we have said that no magnitude is actually infinite, but magnitudes are infinite by division (for the thesis that there are indivisible lines is easily refuted), and therefore they must be potentially infinite. But their being potentially infinite must not be understood as implying that they will (may ?) at some time be actually infinite in the same way as what is potentially a statue will (may ?) at some time be a statue. For *being* has many senses.

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> and the sense in which a thing *is* infinite is the sense in which there *is* a day or a contest, namely by one thing coming into existence after another. For indeed in these cases too we may distinguish between potentiality and actuality: the Olympic games (may) exist both in the sense that they are able to take place and in the sense that they are taking place. (206a 14-25.)

What is clear is that Aristotle begins by claiming that a magnitude such as a length is potentially but not actually 'infinite by division', and then goes on to expound the sense of this claim. It may be noted first that although he writes as if his claim were a conclusion from argument it is in fact nothing of the sort. For what has been said previously is that no magnitude is *infinitely large* (either actually or potentially), but being infinitely large is evidently a case of being 'infinite by addition' (*cf.* 206b 20-27) and so has just been distinguished from being 'infinite by division'. In fact no part of the preceding discussion has any tendency to show that a line cannot be actually infinite by division, and no part of the following discussion seems to offer any *argument* for this claim either. In the end I think it has to be admitted that this is simply a lacuna in Aristotle's account, and a lacuna of some importance, but for the moment let us leave the grounds for Aristotle's claim on one side and consider its elucidation.

Normally, to say that something is potentially ϕ is to imply that it is possible that it should be actually ϕ, and it might seem that Aristotle is intending to deny this implication in the case of infinity. If this were so, he would clearly be using the word 'potentially' in an unusual sense, but in fact it is not the sense of 'potentially' that he mentions as unusual but the relevant sense of 'is': the relevant sense of 'is' is the sense in which a process such as a game may be said to 'be', namely by one thing coming into existence after another. Now in the ensuing discussion it does very often seem that when Aristotle talks of a thing being 'infinite by division' it is indeed a *process* that he has in mind, namely the process of continually dividing the thing (*cf. e.g.*, 206a 27-29). But a process, he here states, may be said to exist actually at any time at which it is going on, and he does *not* deny that the process of continually dividing a finite line can in this way exist actually (206b 12-14). But it seems

equally clear that the sense in which a finite line is potentially infinite by division is again that the process of continually dividing it exists potentially (*i.e.*, *could* exist), and so it turns out in the end that in the sense in which a line is potentially infinite by division it may *also* be actually infinite by division. In that case, what has become of the claim that the infinite is potential but *not* actual?

I think it is clear that what Aristotle mainly had in mind when he first made this claim is just that the process of dividing a line into infinitely many parts is one that cannot be *completed*, and one can indeed understand his contrast with the potential statue in this light. It may well be that what is potentially infinite by division *actually is being* infinitely divided, but it cannot be that it *actually has been* infinitely divided; by contrast a lump of bronze that is potentially a statute may certainly *be becoming* a statue but it may also *have become* a statue, and it is this latter that we mean when we say that it *is* a statue. (And one can see how Aristotle could say that this contrast, which I have just drawn by using different tenses, is a contrast in the relevant senses of 'is'.) So it appears that the claim that Aristotle is making is just the claim that an infinite process is a process that cannot be finished—for, one might ask, how could one come to the end of that which has no end?

Now for the moment we may grant Aristotle's views on infinite *processes*, but it is relevant to recall that infinity is not usually considered to apply only to processes. For instance, it would usually be held that there are infinitely many numbers (as Aristotle has already remarked), and that there are infinitely many points on a finite line, and neither of these cases of infinity appears to be a process in the relevant sense, that is, a series of occurrences which take place one after the other. So when Aristotle claims that these infinities are potential and not actual he cannot easily be understood as claiming that they are processes which cannot be completed. Yet I think that this *is* very close to what he is claiming, for—to consider only the points on a finite line—he does hold that these are dependent on occurrences. In fact he distinguishes between the actual and the merely potential existence of a point in such a way that a point is not said to exist actually until it *has been actualised*, so that there could be an actual infinity of points on a line at one time only if infinitely many points of the line had been actualised by that time. But, assuming that one cannot actualise infinitely many points *all at once*, this must involve the completion of an infinite process of actualisations, and that we have (for the moment) agreed to be impossible. Thus it is that the infinity of points on a line turns out not to be an actual infinity (though it still is a potential infinity, on the ground that there could exist a process of actualising points on the line which was an infinite process.)

The first major difficulty with this view is evidently to understand the notion of actualising a point, and the second is to see why Aristotle was led to the view in the first place. For while it does at first sight seem reasonable to claim that an infinite or unending process cannot ever be completed, there is no such initial plausibility in the claim that an infinite totality could exist only as the result of a completed infinite process. Why, then, does Aristotle wish to deny the actual existence of infinitely many points on a line, and how does he suppose that a point is brought into actual existence? It seems to me that the answer to both these questions is to be found in his treatment of Zeno's problem, to which I now turn.

2 ARISTOTLE ON ZENO

The heart of Zeno's problem may be stated in this way. Suppose we have a body (which we may call Achilles) which moves a finite distance from a starting point to a finishing point (which we may call the tortoise). Now the distance over which Achilles has to travel to reach the tortoise will contain infinitely many points, and any infinite series of such points will mark out the distance into infinitely many discrete parts. Consequently Achilles has to traverse infinitely many parts of the distance, one after the other, before he has traversed the whole distance. But this is as much as to say that before he can reach the tortoise he must have completed a series of infinitely many different tasks, and we have just agreed that this is impossible. So Achilles cannot catch his tortoise after all.

Aristotle's answer to the problem, when stated in this form, is given in *Physics* VIII, chapter 8, 263 a 4-b 9, and this answer relies heavily on the view that a line does not actually contain infinitely many points. Aristotle still maintains (i) that it is impossible to have completed a series of infinitely many actual tasks, and he agrees that traversing an actual part of a line would be an actual task. But he also holds (ii) that an actual part of a line must be bounded by actual points, so Achilles would have to traverse infinitely many actual parts of the line only if he had to pass through infinitely many actual points. However (iii) the points that Achilles passes through cannot be expected to be actual points unless Achilles in some way actualises them as he passes through them, and *merely* passing through a point is not sufficient to actualise it. In fact (iv) a point in a line may be actualised by a body's coming to rest at that point, or by a division actually being made at that point, or indeed by the point's being merely counted; and an infinite series of tasks such as these *cannot* be performed. So finally (v) what we do think Achilles can do, namely to move continuously until he reaches the tortoise, is not after all an infinite series of tasks, but just one task, for it is not actually split up into component tasks. One could specify a task that splits up into an infinite series of component tasks, by requiring Achilles to actualise some infinite series of the intervening points in some way, but that would be specifying a new task and one that Achilles could *not* perform. (For instance one could require Achilles to move $\frac{1}{2}$ the distance in $\frac{1}{2}$ a minute and then rest for $\frac{1}{2}$ a minute; then to move $\frac{1}{4}$ of the distance in $\frac{1}{4}$ of a minute and then rest for $\frac{1}{4}$ of a minute; and so on. Evidently if Achilles could complete this infinite series of tasks he would reach the tortoise in 2 minutes, but Aristotle says—and common sense seems likely to agree with him—that Achilles *cannot* complete *this* series of tasks.)

Now it seems to me that this solution to Zeno's paradox is worth some consideration, for at least at first sight it does give the right results concerning what Achilles can and cannot be expected to do; he can be expected to pass through infinitely many points but he cannot be expected to 'actualise' them, where by 'actualising' a point we mean roughly *doing* something at or to the point which singles it out in some way from its neighbours. That this is the right sort of way to understand the notion of actualising a point may be seen by considering similar Zenonian problems in other areas. For instance, having shown by his well-known arguments that nothing ever moves, Zeno might have completed his dilemma by arguing that nothing ever rests either. For, for Achilles to remain at rest for a full minute he must first remain at rest for $\frac{1}{2}$ a minute, then for the next $\frac{1}{4}$ minute, then for the next $\frac{1}{8}$ minute, and so on. Thus he must have completed an infinite series of tasks of resting before he can be said to have remained at rest for his minute, and that we have said is impossible. But presumably the appropriate Aristotelian reply to this problem is that one does not actualise an instant of time by simply enduring through it; to actualise an instant one has to *do* something *at* it which in some way singles it out from its neighbours, and that condition is not fulfilled by simply remaining at rest during a period in which it falls.

This characterisation of the notion of actualising a point is still very vague, of course, but I am inclined to think that it is at least definite enough for us to see that it does not after all contain the solution to Zeno's problem. First, it is relevant to notice a very obvious objection, namely that when Achilles passes through any intermediate point in his journey to the tortoise then there is of course something which he does at that point, viz he reaches or arrives at it. Now in fact Aristotle goes so far as to argue that this is not after all true. For, he says, if Achilles arrives at the point in his journey then clearly he also leaves the point, for otherwise he would remain there and so never reach the tortoise. But at the time when Achilles has arrived at the point then he is at the point, whereas at the time when he has left the point he is not at it (but beyond it), so the time when he has arrived and the time when he has left are not the same time, and consequently they must be separated by an interval. But during that interval Achilles must evidently be at the point, since he has arrived at it but has not yet left it, and so we may conclude that if Achilles reaches or arrives at any point in his journey then he also rests at it. Hence finally when Achilles is travelling continuously, as we are supposing he is, we must not say that he reaches or arrives at any point which he passes through. (262a 19 - 263a 3.)

Well, this argument is of course fallacious, for if t is the time when Achilles arrives at the point, then all times later than t are times when he has left it, but there is evidently no interval of time between t and all times later than t. But it is interesting

that Aristotle produces the argument, for this does perhaps indicate that he sensed it as an objection to his notion of actualising a point that merely arriving at a point ought to be enough to actualise it. However I do not want to press this objection, which may perhaps seem rather captious, but to turn to the idea of reversing one's direction at a point, which was in fact Aristotle's ultimate target in this argument. For one of the reasons why he argued that arriving at a point, and leaving the point, involve remaining at the point for at least some time, was precisely that he wanted to show that when I reverse my direction at a point then I *must* rest at that point for some interval of time, since in this case we cannot avoid speaking of my arriving and leaving the point. (And he even thinks that observation assures us of this! 262a 17-19.) Now this is clearly a mistake, because, for instance the point which is the limit of a projectile's upward motion and the start of its downward motion, is a point at which the projectile reverses its direction without remaining at the point for any period. But it seems absolutely clear that a point at which one reverses one's direction of motion is a point at which one *does* something that singles that point out from its neighbours, and hence a point which is actualised. And it seems to me that this creates a difficulty for Aristotle's theory of the potential infinite.

3 ZENO'S PROBLEM RESTATED

I shall develop this difficulty with the help of an example, and I take as my example a theory to explain how it is that an ordinary rubber ball bounces on a rigid and unyielding surface. In outline the theory is that when the falling ball strikes the surface, its momentum carries it down yet further, thus compressing the rubber on the side of the ball in contact with the surface and deforming the ball from its natural shape. The ball, however, is elastic, and so resists deformation, with the result that the downward motion of the ball is eventually halted and the deforming process comes to an end. Thereupon the reverse process begins: the ball reverts to its natural shape, exerting pressure on the surface and thereby raising itself once more, and the upward momentum which it thus acquires carries it away from the surface and so brings about a further bounce.

More exactly, let us call the shape which the ball would assume when remaining at rest on the surface its rest shape, and the position it would then assume its rest position. (The rest shape will be slighly deformed, since the ball's own weight will compress the rubber next to the surface to some extent.) Then the theory is that the ball not only accelerates during its downward fall but continues to accelerate downwards even after it has made contact with the surface until it has been deformed into its rest shape and reached its rest position. After it has made contact with the surface the rate of acceleration downwards will of course be diminished, but it will still be positive, for the gravitational attraction responsible for this acceleration will be greater than the opposing upwards force due to the ball's resistance to deformation, until the rest position is reached. And this must be so, for what holds the ball in its rest position when it is at rest is that at that point the downward force of gravity and the upward force due to the ball's resistance to deformation just balance, and therefore at any higher point the force of gravity will be greater than the ball's resistance, and at any lower point the ball's resistance to deformation will be greater than the force of gravity. Consequently the ball's downward momentum will be at its maximum at the time when the ball passes through its rest position on the way down, and by parallel considerations the ball's upward momentum on the next bounce will be at its maximum again at the time when the ball passes through its rest position on the way up. Finally, we assume that the ball does not acquire or lose momentum by fits and starts; all such changes take place strictly continuously.

I have elaborated this theory in some detail because I want to make it quite clear that the theory is an entirely plausible one, and does not involve anything unintelligible. But its point of course is that according to this theory once a ball has started bouncing there is no bounce that is its last bounce. Actually we probably would not say that there had been a *bounce* in the ordinary sense unless the ball has actually left the surface, and it is not a consequence of the theory that every time the ball meets the surface it must leave it again. But it is a consequence of the theory that every time the ball sinks below its rest position it must rise above its rest position again, simply because if it sinks below its rest position it is deformed beyond its rest shape, and its resistance to deformation will then raise it once more to its rest position, and in such a way that it acquires its maximum momentum upwards at the time when it reaches its rest position. Inevitably, then, it will continue to rise further above that position, being carried on by this momentum, until its upward motion is eventually brought to a halt by the deceleration due to gravity. So, as I say, every time the ball sinks below its rest position it will rise above it again, and by parallel considerations every time that it rises above its rest position it will sink below it again, and consequently no oscillation about its rest position will be the last oscillation.

But it would be an error to conclude from this that once our ball has started bouncing (or oscillating) it must continue to do so for ever. *That* is not a consequence of our theory at all. It is an obvious fact of experience that when an ordinary rubber ball is set to bounce its bounces become successively smaller and smaller, taking less and less time each, and this fact is entirely consistent with our theory. Indeed, a moment's reflection is sufficient to show that it is quite consistent with the theory to suppose that with some balls each bounce takes only half as long as the preceding bounce, so that the times taken in successive bounces are in the continued proportion

$$1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \ldots .$$

and all the bounces are completed in a finite time. And of course it is equally consistent to suppose that the times taken in successive bounces are in some different proportion, for instance

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \ldots .$$

so that the bouncing goes on for ever. The theory I have been giving is neutral between these two alternatives, and which kind of series proves to be realised in any given case must wait upon empirical investigation. What I am claiming is simply that the first alternative is a perfectly good possibility, so that it is quite possible that an infinite series of bounces—which is one sort of infinite process—should have been completed.

If all this is admitted, then it seems quite clear that Aristotle's solution to Zeno's problem must be rejected. For Aristotle, wishing to uphold his claim that an infinite series of tasks cannot be completed, was led to argue that Achilles does not traverse infinitely many parts of the line joining his starting point and his goal, on the ground that there are not infinitely many points on that line marking it out into parts. But he was able to deny the (actual) existence of these points only by denying that they had been actualised, for Achilles was not supposed to have done anything at or to the points as he passed them. However this reply is surely not available to the revised version of Zeno's problem that I have just put forward, for the ball's motion is certainly divided into infinitely many parts by the infinitely many points which mark the top of each bounce, and these must surely be admitted to have been actualised in the course of the bouncing. I conclude, then, that Aristotle does not after all have the right solution to Zeno's problem.

4 FURTHER CONSIDERATIONS

But perhaps the argument so far is over hasty, for there is certainly a general reluctance to admit that an infinite series of tasks can be completed, and we have not yet uncovered the source of this reluctance. Indeed it has been argued that such a completion is impossible on logical grounds, but these arguments do not stand up to scrutiny.

The argument which I supplied for Aristotle earlier—*viz.*, that it must be impossible to come to the end of that which has no end—need not detain us long, since it is fairly evident that it rests on an equivocation on the phrase 'come to the end of'. Certainly an infinite series has no last member, and therefore it is indeed impossible to *come to the last member* of the series. But it by no means follows that it is impossible to *finish* the series, *i.e.*, to come to a state in which no member remains outstanding. From the fact that there is no *last* member it does not follow that we cannot perform *every* member.

The other arguments usually advanced here are arguments purporting to show that we cannot give a consistent account of the state which results when every member of the series of tasks has been performed. In the case of the bouncing ball (as so far described) such arguments cannot get started, because it is perfectly clear that in the final state of the system the ball will

simply be at rest on the surface. But suppose we were to harness our ball to some kind of switch mechanism which will be in one of its two states during the performance of one bounce and in the other during the next bounce. Then, it will be argued, the switch cannot consistently be supposed to be in either of its two states when the infinite series of bounces has been completed. But the reply to such arguments is generally that there is no such inconsistency as they allege, and if there is anything puzzling here it is just that *neither* of the two incompatible end-states is inconsistent with the initial specification.

In more detail, let us begin with an entirely fanciful switch mechanism as an example, to avoid the difficulty that any ordinary switch mechanism will have to be able to operate at arbitrarily high speeds. Let us suppose that our bouncing ball is white during its first bounce and changes colour from white to black and back again at each bounce—in fact let us specify that it changes colour exactly at the moment that it passes through its rest position on the upward journey, and that it changes colour at no other time. Now I agree that if the original account of the bouncing ball was intelligible then this further specified account is also intelligible. But, it will be argued this further specified account has become unintelligible. For since our ball can change colour only between white and black, it must be either white or black when the bouncing is finished. But if it is white then there were an odd number of bounces in all, and if it is black then there were an even number of bounces in all, and yet neither of these is possible since only a finite number can be odd or even, and the number of bounces was supposed not to be a finite number. However a little thought will show that this argument is guilty of an illicit conversion: from the initial specification it can indeed be deduced that if the ball bounced for an odd number of times it would end as white, but it cannot be deduced that if the ball ends as white then it bounced for an odd number of times. In fact the position is that we have been given enough information to determine the colour of the ball during all the time that it is bouncing, but we have not been given enough to determine its colour when it has finished; or, put in another way, our information will give us the colour of the ball at any time when it has completed only a finite number of bounces, but it does not tell us what happens when the ball has completed infinitely many bounces.

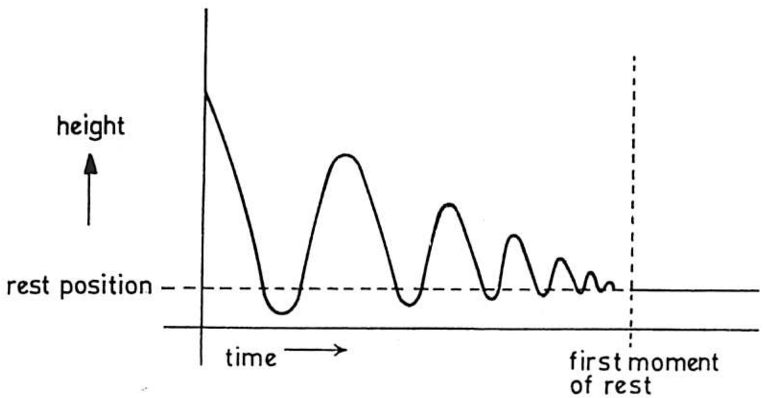
To see this more clearly, let us attempt to deduce its final colour. If we consider the ball's first moment of rest—and I mean the first moment such that the ball is at its rest position at that moment and at all succeeding moments—then it is clear that the ball does not change colour at that moment. For we specified that it changed colour only when *passing through* its rest position (on the upward journey), and it is not passing through its rest position at its first moment of rest, but coming to rest at it. So, one is inclined to say, if it does not change colour at that moment we ought to be able to infer that it will be the same colour then as it was *at the previous moment*. But of course the trouble is that there is no such thing as the previous moment, for quite generally no two moments are *next* moments (since between any two moments there are others). The most that one could hope to infer, then, is that the ball will be the same colour at its first moment of rest as it was at *all* preceding moments within a sufficiently short interval. But here the trouble is that there is no immediately preceding interval that is short enough to ensure that the ball remains the same colour throughout it, and all methods of trying to deduce the ball's colour at its first moment of rest must therefore fail. Consequently we are at liberty to make what assumptions we like about it. We may take it to be black and no contradiction will result, and we may take it to be white and no contradiction will result. Where, then, is the difficulty?

If there is any difficulty it is just that we mistakenly suppose that the initial specification *must* suffice to determine the ball's colour at all subsequent times, and this is particularly liable to be felt as a difficulty if we are considering not my fanciful example of colour-changing but something that is recognisably a *causal mechanism*. Suppose, for instance, we imagine that whenever the pressure of the ball on the surface is greater than its pressure when at rest on the surface it closes a circuit permitting current to flow which moves a genuine switch from one of its two positions to the other. Then to the question where the switch is at the end of the bouncing we must reply as before that we have not been given enough information to determine the matter, but in this case the reply is apt to seem quite

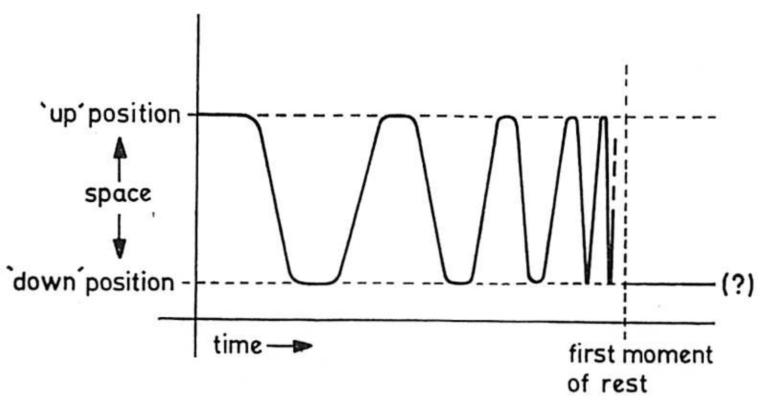
mysterious. For in this case the lack appears not to be in the information we have been given but in the nature of things: if the final position of the switch is not deducible from this information, then it is not determined by the causal laws which govern the operation of the switch, and so whatever happens there will have been a breakdown in causal determinism. For we may suppose that we are given the initial positions of the ball and the switch at a certain time, and that we possess a theory concerning the subsequent behaviour of the ball which allows us to predict its position and momentum at all subsequent times. We also have a further theory connecting the behaviour of the ball with the behaviour of the switch, which we would say makes the behaviour of the switch wholly dependent on that of the ball—at least, the behaviour of the switch is not determined by anything else than by the behaviour of the ball. But we now find that although we can predict the position of the switch at all times before the ball's first moment of rest, we cannot predict its position at that moment or after it. It seems clear that our inability to predict is not due to a lack of knowledge of the initial state of the system, and it does not seem at all plausible to say that it is due to a lack of knowledge of the causal laws governing the system, and if this is granted it then follows that the system is not deterministic in a Laplacian sense: whatever the final position of the switch turns out to be, it is uncaused.

My conclusion is that there is not in general any *logical* impossibility in the idea of completing an infinite series of tasks; this is actually achieved by Achilles when he catches his tortoise, and there is no difficulty in supposing it achieved by the bouncing ball. On the other hand there are cases where the idea that such an infinite series may be completed runs into conflict with our belief in causal determinism (and, incidentally, with our belief that every mechanism has a highest speed of operation), and it seems to be these cases that lead people to say that the completion is impossible. I shall end by making just one suggestion as to the fundamental difference between the cases where this conflict does arise and where it does not, though I have no space here to explore the suggestion in any detail.

If we draw a graph to represent the motion of our bouncing ball it will have the general shape



and this curve is continuous, despite the fact that there are infinitely many oscillations before the first moment of rest. However if we now consider the motion of the troublesome switch, we may imagine this to have a pointer that is in 'up' position during odd-numbered bounces and in 'down' position during even-numbered bounces, and moves ever more rapidly between the two. In that case a graph representing its motion would have the general shape



and now, wherever we suppose the pointer to be at and after

the first moment of rest, the curve *must* be discontinuous at that point. There is no logical impossibility in the idea of a motion that is represented by such a discontinuous curve, but I think we do find it repugnant to our idea of causality in nature, for the state of the switch at and after the first moment of rest cannot be smoothly connected with its previous states. When a completed infinite process would involve such a discontinuity in nature, then—and perhaps only then—we tend to regard it as impossible.